Assignment 10

Hand in no. 1, 2, 4 and 7 by Nov 28.

1. Determine which of the following sets are dense, open dense, nowhere dense, of first category and residual in $\mathbb{R}$ (you may draw a table):
   (a) $A = \{n/2^m : \ n, m \in \mathbb{Z}\}$,
   (b) B, all irrational numbers,
   (c) $C = \{0, 1, 1/2, 1/3, \cdots\}$,
   (d) $D = \{1, 1/2, 1/3, \cdots\}$,
   (e) $E = \{x : \ x^2 + 3x - 6 = 0\}$,
   (f) $F = \cup_k (k, k+1), k \in \mathbb{N}$,

2. Determine which of the following sets are dense, open dense, nowhere dense, of first category and residual in $C[0,1]$ (you may draw a table):
   (a) $\mathcal{A}$, all polynomials whose coefficients are rational numbers,
   (b) $\mathcal{B}$, all polynomials,
   (c) $\mathcal{C} = \{f : \ \int_0^1 f(x)dx \neq 0\}$,
   (d) $\mathcal{D} = \{f : \ f(1/2) = 1\}$.

3. Use Baire Category Theorem to show that transcendental numbers are dense in the set of real numbers.

4. A point p in a metric space X is called an *isolated point* if there is an open set G such that $G \cap X = \{p\}$, that is, $\{p\}$ is open. A set E in X is a *perfect set* if it is closed and contains no isolated points.
   (a) For each x in the perfect set E, there exists a sequence in E consisting of infinitely many distinct points converging to x.
   (b) Every perfect set is uncountable in a complete metric space.

5. Let f be a real-valued function on $\mathbb{R}$. Define the oscillation of f at x to be $\omega_f(x) = \lim_{\delta \to 0^+} \omega_f(x, \delta)$ where
$$\omega_f(x, \delta) = \sup\{|f(y) - f(z)| : y, z \in (x - \delta, x + \delta)\} .$$
   (a) The set $D = \{x : \ \omega_f(x) \geq \rho\}$ is closed for all $\rho > 0$.
   (b) Show that the set of discontinuous point of f is given by $\bigcup_n D_n$ where $D_n = \{x : \omega_f(x) \geq 1/n\}$.
   (c) Show that we cannot find a function which is discontinuous exactly at all irrational numbers.

6. Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$.
   (a) Show that $\|x\| \leq C\|x\|_2$ for some C where $\|\cdot\|_2$ is the Euclidean metric.
   (b) Deduce from (a) that the function $x \mapsto \|x\|$ is continuous with respect to the Euclidean metric.

(c) Show that the inequality $\|x\|_2 \le C'\|x\|$ for some C' also holds. Hint: Observe that $x \mapsto \|x\|$ is positive on the unit sphere $\{x \in \mathbb{R}^n : \|x\|_2 = 1\}$ which is compact (that is, closed and bounded).

(d) Establish the theorem asserting any two norms in a finite dimensional vector space are equivalent.

7. Let P be the vector space consisting of all polynomials. Show that we cannot find a norm on P so that it becomes a Banach space.

8. Let $\mathcal{F}$ be a subset of $C(X)$ where X is a complete metric space. Suppose that for each $x \in X$, there exists a constant M depending on x such that $|f(x)| \le M, \ \forall f \in \mathcal{F}$. Prove that there exists an open set G in X and a constant C such that $\sup_{x\in G} |f(x)| \le C$ for all $f \in \mathcal{F}$. Suggestion: Consider the decomposition of X into the sets $X_n = \{x \in X : |f(x)| \le n, \ \ \forall f \in \mathcal{F}\}$.